Review Problems

Introduction

Below you will find a compilation of some extra problems which you may find useful in reviewing for your quizzes and exams.

There are four sections of problems. Within each section, the problems are organized randomly. Some problems are especially challenging so don't despair if you find yourself stumped by a portion. The complete solutions appear after the four sections of problems.

Integrals and related problems

1. Determine whether $\int_0^2 \frac{dx}{4x-5}$ is improper. If improper either evaluate, or prove that the integral is divergent.

2. True or False: $\int_{-4}^{4} \frac{dx}{5x^{1/3}-4}$ converges by comparison to $\int_{-4}^{4} \frac{dx}{x^{\frac{1}{3}}}$.

3. Find the arc length of $y = \cosh x$ on the interval $0 \le x \le 1$.

4. Evaluate the integral: $\int \tan^4 x\, dx$.

5. Evaluate the integral: $\int \sqrt{2x - x^2}\, dx$.

6. (a) Evaluate the integral $\int \frac{dx}{x^2(x+2)}$.

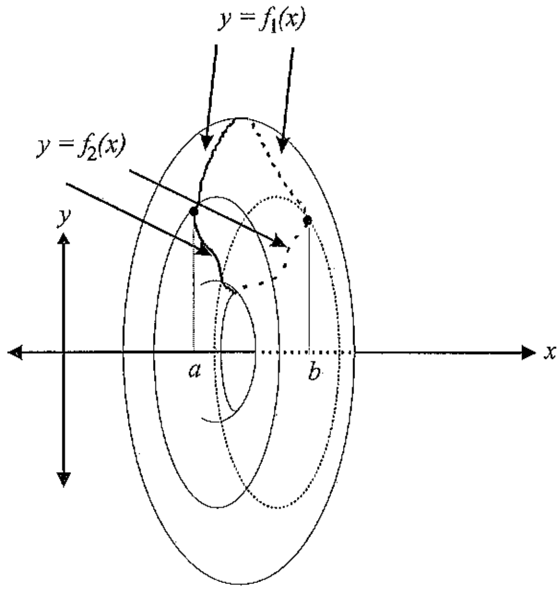
   (b) Evaluate $\int_1^{\infty} \frac{dx}{x^2(x+2)}$ or show that it is divergent.

7. Find the length of the curve defined by:

$$y = \frac{1}{x^2}, \quad 0 < x \le 1.$$

8. Find the partial fractions decomposition (including the values of A, B, etc.) of $\frac{x^3+2x}{x^3+1}$.

9. Below is pictured a surface of revolution generated by rotating the curves $y = f_1(x)$ and $y = f_2(x)$ around the x-axis.



Find a formula for the surface area of this surface, involving $f_1(x), f_2(x)$, and their derivatives.

10. Integrate: $\displaystyle\int \frac{\sin x}{\cos^{101} x}\,dx$

11. Which of the curves below has *both* of the following properties:

   - its length is *infinite.*
   - the area beneath it and above the x-axis is *finite.*

   (a) $y = \dfrac{1}{\sqrt{x}\cdot|\ln x|}, \quad 0 < x \le e^{-1}.$

   (b) $y = \dfrac{1}{|\ln x|}, \quad 1 < x < \infty.$

   (c) $y = \dfrac{1}{\sqrt{x}}, \quad e^{-1} < x \le 1.$

   (d) $y = \dfrac{1}{x^2}, \quad 0 < x \le e^{-1}.$

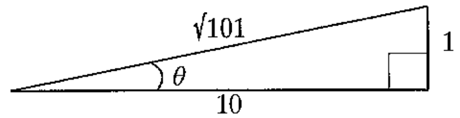
   (e) None of the above.

Sequences and Series

1. Does the following series converge or diverge. (Justify.)

$$\sum_{n=0}^{\infty} \frac{(1,000,000)^n}{n!}$$

2. Below you may use the formula $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. Consider the following figure:



   (a) Using the above, find an expression for θ in terms of an infinite sum.

   (b) Find an approximation for θ with an error less than 0.00001. (You don't need to simplify any fractions that you may have, and needn't express your answer using decimals.)

3. Find a power series representation for each of the following. State the *radius* of convergence in each case.

   (a) $-\ln(1-x)$

   (b) $\ln(1+x)$

   (c) $\ln\left(\frac{1+x}{1-x}\right)$

4. Find the Maclaurin series for $\ln(1-x^3)$. What is the corresponding Taylor polynomial, $T_7(x)$ about $x=0$?

5. Give an example of each of the following. (No explanation required.)

   (a) An infinite sum whose convergence can be decided by the ratio test.

   (b) An infinite sum whose convergence can be decided by the root test.

   (c) A sequence that is bounded above but diverges.

6. (a) Does $\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^{1.5}}$ converge or diverge?

   (b) Does $\sum_{n=100{,}000}^{\infty} \frac{1}{n(\ln n)^{1.5}}$ converge or diverge?

7. Determine whether or not the given sum converges. Find its value if it does. (Justify)

   (a) $\sum_{n=1}^{\infty} \left[\sin\left(\frac{n+1}{n}\right) - \sin\left(\frac{n+2}{n+1}\right)\right]$

   (b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

8. For (a) and (b) determine whether the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, absolutely convergent, or divergent.

   (a) $a_n = \dfrac{n^2 - n + 2}{\sqrt[4]{n^{10} + n^5 + 3}}$

   (b) $a_n = (-1)^n \dfrac{1 + e^{-n}}{n}$

9. Find the Taylor series of the function $f(x) = \dfrac{1}{\sqrt{x}}$ about the point $a = 1$.

10. Determine whether each of the following diverges or converges. (Justify.)

    (a) $.9 - .99 + .999 - .9999 + .99999 - .999999 + \cdots$

    (b) $\dfrac{1}{2^2} + \dfrac{2}{3^2} + \dfrac{3}{4^2} + \cdots$

    (c) $\sum_{n=1}^{\infty} (\dfrac{n^2 + 2n}{n^3 + 1} - \dfrac{1}{2})^n$

11. Find the radius of convergence for:

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} 3^{3n} x^n.$$

12. For (a) and (b) determine if the sequence $\{a_n\}$ converges. If it does, find the limit.

    (a) $a_n = \dfrac{1}{n^{-\ln n}}$

    (b) $a_n = \sqrt{\dfrac{(1+n)n}{\sin n + n^2}}$

13. Obtain the Taylor series of $1 - \sin^2 x$ about $x = 0$.
    [Hint: trigonometric identities.]

Differential Equations

1. Find the general solution of the ODE:

$$y' = \cos^2 y \cdot \ln x$$

2. Solve $y' + \cos x \cdot y = \sin x \cdot \cos x$.

3. Find the general solution of $y' = \dfrac{y}{x} + 2$.

4. Sketch a direction field for $\frac{dy}{dt} = \frac{y}{t}$. Then for each initial condition below, graph a solution curve for $t \geq 1$ on the direction field which satisfies the condition.

   (a) $y(1) = 0$.

   (b) $y(1) = 1$.

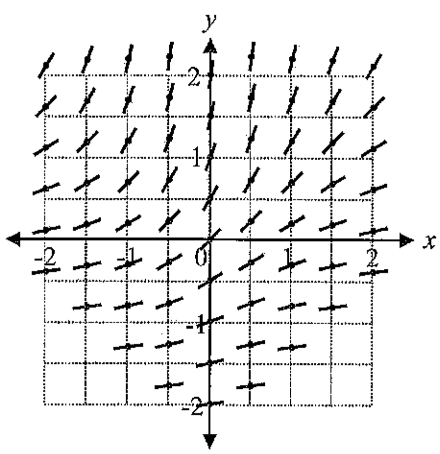
5. True or false: The families of curves $x = ky^2$ and $\frac{1}{2}x^2 + y^2 = c$ (c and k are constants) are orthogonal trajectories. (Justify.)

6. Solve:
$$y'' - 2y' - 3y = 0; \quad y(0) = 3, \quad y'(0) = 1.$$

7. Solve:
$$y'' - 2y' + 5y = 0.$$

8. Solve:
$$y'' - 6y' + 9y = 0; \quad y(0) = 1, y(1) = e^4 + e^3.$$

9. Below is pictured a direction field for a differential equation of the form
$$y' = f(x, y).$$



Which of the following best describes the function $f(x, y)$?

(a) $\frac{y}{x} + e^{\frac{y}{x}}$

(b) $\frac{e^y}{1 + x^2}$

(c) $\csc x$

(d) $\sin y$

10. Consider the linear differential equation

$$y' \cdot \cos x = y \cdot \sin x + e^x \cos x$$

   (a) Which of the following is an *integrating factor* for the differential equation:
      i. $I(x) = e^{\int e^x \cos x \, dx}$
      ii. $I(x) = \sin x$
      iii. $I(x) = e^{-\ln|\cos x|}$
      iv. $I(x) = e^{\cos x}$
      v. $I(x) = \cos x$
      vi. $I(x) = e^{-\cos x}$
      vii. None of the above.

   (b) Find the general solution to the above differential equation.

11. Find the solution of $\dfrac{d^2y}{dx^2} = xy; \quad y(0) = 1, \quad \dfrac{dy}{dx}(0) = 0.$

Complex Numbers

1. Let $z = 1 - i\sqrt{3}$.
   (a) Find $|z|$.
   (b) Find $\arg z$.
   (c) Find z^5 . [Hint DeMoivre, or Euler.]

2. (a) Find all solutions to the equation $x^2 - 2x + 5 = 0$.
   (b) For each solution x, write x^2 and $\dfrac{1}{x}$ in the form $a + ib$.

3. (a) Solve $x^6 = -1$.
   (b) Factor $x^6 + 1$ over $\mathbf{C}$.
   (c) Factor $x^6 + 1$ over $\mathbf{R}$.

Solutions to Review Problems

Integrals, etc.

1. Let's integrate using the substitution

$$\left\langle \begin{array}{c} u = 4x - 5 \\ u + 5 = 4x \\ x = \frac{u+5}{4} \end{array} \;;\; dx = \frac{du}{4} \;;\; \begin{array}{ccc} x = 2 & \Rightarrow & u = 3 \\ x = 0 & \Rightarrow & u = -5 \end{array} \right\rangle$$

to get the integral: $\displaystyle\int_{-5}^{3} \frac{\frac{du}{4}}{u} = \frac{1}{4}\int_{-5}^{3} \frac{du}{u}$. At this point, we need to be careful because the singularity $u = 0$ occurs *between* the endpoints -5 and 3. A common mistake is to formally take the anti-derivative at this point, and then plug in the end-points to get the answer: $\frac{1}{4}(\ln|3| - \ln|-5|) = \frac{1}{4}\ln\frac{3}{5}$. But this is a **fallacy** because the integral is *improper* due to the point $u = 0$ at which the integrand $\frac{1}{u}$ is undefined!

The correct way to evaluate the integral is by breaking it up into a sum of improper integrals, each with only an *endpoint singularity* as follows:

$$\frac{1}{4}\int_{-5}^{3} \frac{du}{u} = \frac{1}{4}\int_{-5}^{0} \frac{du}{u} + \frac{1}{4}\int_{0}^{3} \frac{du}{u}$$

where the first summand has a right endpoint singularity, and the second summand has a left endpoint singularity. To show that $\frac{1}{4}\int_{-5}^{3} \frac{du}{u}$ *diverges*, it is enough to show that one of the above diverges. E.g., choose the second. By definition of an improper integral of type II we have

$$\int_{0}^{3} \frac{du}{u} = \lim_{t\to 0^+} \left[\int_{t}^{3} \frac{du}{u}\right]$$

$$= \lim_{t\to 0^+} \left[\ln u\Big|_t^3\right] = \lim_{t\to 0^+}[\ln 3 - \ln t] = \ln 3 - \lim_{t\to 0^+} \ln t = +\infty$$

Therefore, the integral is divergent.

2. FALSE.

Heuristic answer(**WRONG**): $\displaystyle\int_{-4}^{4} \frac{dx}{5x^{1/3} - 4}$ converges because it behaves much like $\displaystyle\int_{-4}^{4} \frac{dx}{x^{\frac{1}{3}}}$ which converges.

Surprising rigorous answer: Use the substitution

$$\left\langle \begin{array}{c} u = 5x^{\frac{1}{3}} - 4 \\ \frac{u+4}{5} = x^{\frac{1}{3}} \\ x = \frac{1}{5^3}(u+4)^3 \end{array} \;;\; dx = \frac{3}{5^3}(u+4)^2\, du \;;\; \begin{array}{ccc} x = 4 & \Rightarrow & u = 5\cdot 4^{\frac{1}{3}} - 4 \\ x = -4 & \Rightarrow & u = -5\cdot 4^{\frac{1}{3}} - 4 \end{array} \right\rangle$$

to get the integral

$$\int_{-5\cdot4^{\frac{1}{3}}-4}^{5\cdot4^{\frac{1}{3}}-4}\frac{\frac{3}{5^3}(u+4)^2\,du}{u}=\frac{3}{5^3}\left[\int_{-5\cdot4^{\frac{1}{3}}-4}^{0}\frac{(u+4)^2\,du}{u}+\int_{0}^{5\cdot4^{\frac{1}{3}}-4}\frac{(u+4)^2\,du}{u}\right]$$

To show that this integral *diverges* we need to show that one of the improper integrals, say $\int_0^{5\cdot4^{\frac{1}{3}}-4}\frac{(u+4)^2\,du}{u}$ diverges, and to show this, it's enough to show that $\int_0^1\frac{(u+4)^2\,du}{u}$ diverges. This is done by comparing to the divergent integral $\int_0^1\frac{du}{u}$ by using the fact that $(u+4)^2>1$ for $u\geq 0$.

3. The arc length of the hyperbolic cosine:

$$y=\cosh x\Longrightarrow y'=\sinh x$$

$$\Longrightarrow\sqrt{1+(y')^2}=\sqrt{1+\sinh^2 x}=\sqrt{\cosh^2 x}=\cosh x$$

The arc length of the curve given by $y=\cosh x$ is therefore computed via the formula:

$$\mathcal{L}=\int_0^1\sqrt{1+(y')^2}\,dx=\int_0^1\cosh x\,dx$$

$$=\sinh x\Big|_0^1=\sinh(1)-\sinh(0)=\sinh(1)$$

where we've used the fact that $\frac{d}{dx}\sinh x=\cosh x$, and that $\sinh(0)=0$. Finally, using the formula

$$\sinh x=\frac{e^x-e^{-x}}{2}$$

we get the answer: $\boxed{\sinh(1)=\frac{1}{2}(e-\frac{1}{e})}$.

4. The main point here is to use the trigonometric identity $\tan^2 x=\sec^2 x-1$:

$$\int\tan^4 x\,dx=\int\tan^2 x\tan^2 x\,dx=\int\tan^2 x(\sec^2 x-1)\,dx$$

$$=\int\tan^2 x\sec^2 x\,dx-\int\tan^2 x\,dx=\int\tan^2 x\sec^2 x\,dx-\int(\sec^2 x-1)\,dx$$

$$=\int\tan^2 x\sec^2 x\,dx-\int\sec^2 x\,dx+\int dx=\langle u=\tan x\ ;\ du=\sec^2 x\,dx\rangle-\tan x+x$$

$$=\int u^2\,du-\tan x+x=\frac{u^3}{3}-\tan x+x+C=\boxed{\frac{1}{3}\tan^3 x-\tan x+x+C}$$

5. Here we want to complete the square, so that we have

$$2x-x^2=-(x^2-2x)=-([x^2-2x+1]-1)=-([x-1]^2-1)=1-[x-1]^2\ ,$$

and then apply the trigonometric substitution $x-1=\sin\theta$ to integrate using the identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$:

$$\int \sqrt{2x-x^2}\,dx = \int \sqrt{1-[x-1]^2}\,dx = \langle\ x-1=\sin\theta\ ;\ dx=\cos\theta\ d\theta\ \rangle$$
$$= \int \cos\theta\cdot\cos\theta\ d\theta = \int \cos^2\theta\ d\theta = \int \frac{1+\cos 2\theta}{2}\ d\theta$$
$$= \frac{\theta}{2}+\frac{\sin 2\theta}{4}+C = \frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}+C$$

where in the last step we used the identity $\sin 2\theta = 2\sin\theta\cos\theta$. Back-substitute using the fact that $x-1=\sin\theta$ so that $\theta=\sin^{-1}(x-1)$ and after setting up an appropriate triangle

$$\cos\theta = \sqrt{1-[x-1]^2} = \sqrt{2x-x^2}$$

so that our integral becomes: $\boxed{\frac{1}{2}\sin^{-1}(x-1)+\frac{1}{2}(x-1)\sqrt{2x-x^2}+C}$

6. (a) First look for a partial fractions decomposition of the form

$$\frac{1}{x^2(x+2)} = \frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+2}$$

and multiply both side of the equation by the denominator of the left side, then solving for A, B, and C to get $A=\frac{1}{2}, B=-\frac{1}{4}$, and $C=\frac{1}{4}$. Thus we can now integrate the partial fraction:

$$\int \frac{1}{x^2(x+2)}\,dx = \frac{1}{2}\int\frac{dx}{x^2} - \frac{1}{4}\int\frac{dx}{x} + \frac{1}{4}\int\frac{dx}{x+2}$$
$$= -\frac{1}{2x} - \frac{1}{4}\ln|x| + \frac{1}{4}\ln|x+2| + C = \boxed{-\frac{1}{2x}+\frac{1}{4}\ln\left|\frac{x+2}{x}\right|+C}$$

(b) Now take the limit as $x\to\infty$ to find the value of the improper integral:

$$\int_1^\infty \frac{1}{x^2(x+2)}\,dx = -\frac{1}{2x}+\frac{1}{4}\ln|\frac{x+2}{x}|\Big|_0^{R\to\infty}$$
$$= \lim_{R\to\infty}[-\frac{1}{2R}+\frac{1}{4}\ln|\frac{R+2}{R}|] - [-\frac{1}{2}+\frac{1}{4}\ln|\frac{1+2}{1}|]$$
$$= [0+\frac{1}{4}\ln(\lim_{R\to\infty}\frac{R+2}{R})]+\frac{1}{2}-\frac{1}{4}\ln 3 = \frac{1}{4}\ln 1+\frac{1}{2}-\frac{1}{4}\ln 3 = \boxed{\frac{1}{2}-\frac{\ln 3}{4}}$$

7. This is a trick question. As $y=\dfrac{1}{x^2}$ has a vertical asymptote at $x=0$ conclude that $\boxed{\text{the length of the curve is infinite}}$.

8. First we need to divide the numerator by the denominator. This can be done by polynomial long division or by using the trick:

$$\frac{x^3+2x}{x^3+1} = \frac{(x^3+1)-1+2x}{x^3+1} = 1 + \frac{2x-1}{x^3+1}.$$

The next step is to factor the denominator. This is done by guessing a root and then using synthetic division (or any other method you like for factoring once you know a root). Start by guessing $\pm 1, \pm 2, \ldots$ etc. The guess "-1" works as $(-1)^3+1=0$. Then factor to get $x^3-1=(x+1)(x^2-x+1)$. Now x^2-x+1 is *irreducible* because its discriminant $b^2-4ac=1-4=-3$ is negative, so we cannot factor the quadratic over the real numbers. Thus we look for a partial fractions decomposition of the form

$$\frac{2x-1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

Multiplying by the denominator of the left side, doing lots of algebra, and solving for A, B, and C we get: $A=-1, B=1$, and $C=0$ so that the answer is:

$$\frac{x^3+2x}{x^3+1} = 1 + \frac{2x-1}{(x+1)(x^2-x+1)} = \boxed{1 - \frac{1}{x+1} + \frac{x}{x^2-x+1}}$$

9. The main point of this problem is to understand how the formula for area of a surface of revolution works. As the surface is the union of the surface generated by rotating $y=f_1(x)$ and the surface generated by $y=f_2(x)$ around the x-axis we get:

$$\boxed{A = \int_a^b 2\pi f_1(x)\sqrt{1+[f_1'(x)]^2}\,dx + \int_a^b 2\pi f_2(x)\sqrt{1+[f_2'(x)]^2}\,dx}$$

10. There are many ways to do this problem. Here's one way:

$$\begin{aligned}
&\int \frac{\sin x}{\cos^{101} x}\,dx \\
&= \int \frac{1}{\cos^{100} x}\cdot\frac{\sin x}{\cos x}\,dx = \int \sec^{100} x \cdot \tan x\,dx = \int \sec^{99} x \cdot (\sec x \tan x\,dx) \\
&= \langle u = \sec x\ ,\ du = \sec x \tan x\,dx \rangle = \int u^{99}\cdot du = \frac{u^{100}}{100} + C \\
&= \boxed{\frac{\sec^{100} x}{100} + C}
\end{aligned}$$

11. (a)(ANSWER) $y = \dfrac{1}{\sqrt{x}\cdot|\ln x|}, \quad 0 < x \le e^{-1}.$

This answer works because the curve is infinite as it has a vertical asymptote at $x=0$, and has finite area by comparison with $\dfrac{1}{\sqrt{x}}$.

(b) can't work because $\int_1^\infty \frac{dx}{\ln x}$ diverges.

(c) can't work because the curve is finite.

(d) doesn't work since $\int_0^{e^{-1}} \frac{dx}{x^2}$ diverges.

Sequences and Series

1. Let's use the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(1,000,000)^{n+1}}{(n+1)!} \cdot \frac{n!}{(1,000,000)^n} = \frac{1,000,000}{n+1} \longrightarrow 0$$

as $n \to \infty$. As $0 < 1$ we conclude that the series $\boxed{\text{converges}}$.

2. (a) $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{10}$. Therefore, $\theta = \tan^{-1}\left(\frac{1}{10}\right)$. Plug this into the power series expansion for $\tan^{-1} x$ to get:

$$\boxed{\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{10}\right)^{2n+1}}$$

(b) The sum above satisfies the hypotheses of the *alternating series test*. Thus we can use the *alternating sum estimation theorem* to deduce that the error we get by summing up the first N terms, is less than the absolute value of the $(N+1)$'st term. So just start writing out the series, and then keep only the part of the series whose terms are bigger than, or equal to (in absolute value) to 0.00001.

So expanding out the series, we get:

$$\theta = \frac{1}{10} - \frac{1}{3\cdot 10^3} + \frac{1}{5\cdot 10^5} + \cdots = \frac{1}{10} - \frac{1}{3,000} + \frac{1}{500,000} - \cdots$$

Notice that $\frac{1}{500,000} = 2 \cdot \frac{1}{1,000,000} = 0.000002$ which is already smaller than 0.00001, so we can stop with just the *first two* terms to get the approximation:

$$\boxed{\theta \approx \frac{1}{10} - \frac{1}{3,000} = \frac{299}{3,000} \text{ radians}}$$

3. (a), (b), and (c) all have the same radius of convergence, $R = 1$. The power series representations can be obtained as follows:

(a) *Integrate* the geometric expansion to get:

$-\ln(1-x) + C = \int \frac{dx}{1-x} = \int \sum_{n=0}^{\infty} x^n \, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$. Plug in $x = 0$ to get $-\ln(1-0) + C = 0$ so $0 + C = 0$. Therefore:

$$\boxed{-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}}$$

(b) Substitute $y = -x$ in the above to get:
$-\ln(1+x) = -\ln(1-y) = \sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n+1}$. Now take the negatives of both sides to get:

$$\ln(1+x) = -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{n+1}}{n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n}x^{n+1}}{n+1}}$$

(c)

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n}x^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \Big[(-1)^n + 1\Big] \frac{x^{n+1}}{n+1}$$

Notice that:

$$[(-1)^n + 1] = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

So keep only the *even* part of the series to get the solution:

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n \text{ even}} [2]\frac{x^{n+1}}{n+1} = \langle n \to 2k \rangle = \boxed{\sum_{k=0}^{\infty} \frac{2x^{2k+1}}{2k+1}}$$

4. The easiest way to do this problem is to first memorize the expansion $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)} x^{n+1}$, or at least to know how to derive this identity by integrating the geometric series for $\frac{1}{1+x}$. Now let $y = -x^3$ and use the expansion for $\ln(1+y)$ to get $\ln(1-x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(-x^3)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{n+1} x^{3n+3} =$ $\boxed{-\sum_{n=0}^{\infty} \frac{1}{n+1} x^{3n+3}}$.

   For T_7 keep only the terms involving powers of x of degree 7 or less to get: $\boxed{T_7(x) = -x^3 - \frac{1}{2}x^6}$

5. Here are some of the multitudes of possible answers:

   (a) $\sum_{n=0}^{\infty} \frac{1}{n!}$

(b) $\sum_{n=0}^{\infty} \frac{1}{n^n}$

(c) $\{-n\}_{n=1}^{\infty}$

6. (a) $\int_{e^2}^{\infty} \frac{dx}{x(\ln x)^{1.5}} = \langle u = \ln x \; ; \; du = \frac{dx}{x} \rangle = \int_2^{\infty} \frac{du}{u^{1.5}}$
which converges since $p = 1.5 > 1$.

(b) $\sum_{n=100,000}^{\infty} \frac{1}{n(\ln n)^{1.5}}$ converges by using the integral test with $\int_{100,000}^{\infty} \frac{dx}{x(\ln x)^{1.5}}$ which converges by part (a).

7. (a) $\sum_{n=1}^{\infty} \left[\sin\left(\frac{n+1}{n}\right) - \sin\left(\frac{n+2}{n+1}\right) \right]$ is a *telescoping series* whose partial sums (after crossing out all the middle terms) are given by:

$$s_N = \sin(2) - \sin\left(\frac{N+2}{N+1}\right).$$

Taking the limit of this as $N \to \infty$ we get: $\boxed{\sin(2) - \sin(1)}$.

(b) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges by a simple application of the *ratio test.* To evaluate, notice that powers of $\frac{1}{2}$ appear in each term of the infinite sum. The trick is to let $x = \frac{1}{2}$ in the series, and manipulate it, trying to see what function in x it is equal to:
$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x\frac{d}{dx} \sum_{n=0}^{\infty} x^n = x\frac{d}{dx}\frac{1}{1-x} = x\frac{1}{(1-x)^2}$. Now replace x by $\frac{1}{2}$ to get: $\frac{1}{2}\frac{1}{(1-\frac{1}{2})^2} = \boxed{2}$.

8. (a) Divergent by limit comparison with $\sum \frac{1}{\sqrt{n}}$.

(b) Conditionally convergent. It is convergent by the alternating series test, but is *not* absolutely convergent by comparison to $\sum \frac{1}{n}$.

9. The answer is $\boxed{1 + \sum_{n=1}^{\infty} (-\frac{1}{2})^n \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} (x-1)^n}$. To see this, use the binomial series on:
$f(x) = \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{1+(x-1)}} = (1 + (x-1))^{-\frac{1}{2}}$,
or just use the formula for a Taylor series involving lots of derivatives.

10. (a) $.9 - .99 + .999 - .9999 + .99999 - .999999 + \cdots$
$= \sum_{n=1}^{\infty} (-1)^{n-1} \left[1 - \left(\frac{1}{10}\right)^n\right]$ *diverges* by the test for divergence.

(b) $\frac{1}{2^2}+\frac{2}{3^2}+\frac{3}{4^2}+\cdots=\sum_{n=1}^{\infty}\frac{n}{(n+1)^2}$ *diverges* by limit comparison with $\sum\frac{1}{n}$.

(c) For $\sum_{n=1}^{\infty}(\frac{n^2+2n}{n^3+1}-\frac{1}{2})^n$ use the root test:
$\sqrt[n]{|a_n|}=\left|\frac{n^2+2n}{n^3+1}-\frac{1}{2}\right|=\left|\frac{n^{-1}+2n^{-2}}{1+n^{-3}}-\frac{1}{2}\right|\longrightarrow\frac{1}{2}<1$ as $n\to\infty$. This implies *convergence.*

11. Use the ratio test with $a_n=\frac{(n!)^3}{(3n)!}3^{3n}x^n$. We have:

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{([n+1]!)^3 3^{3[n+1]}|x|^{[n+1]}}{(3[n+1])!}\cdot\frac{(3n)!}{(n!)^3 3^{3n}|x|^n}$$
$$=\frac{(n+1)^3(n!)^3\cdot 3^3 3^{3n}\cdot|x||x|^n}{(3n+3)\cdot(3n+2)\cdot(3n+1)\cdot[(3n)!]}\cdot\frac{(3n)!}{(n!)^3 3^{3n}|x|^n}=\frac{27(n+1)^3|x|}{(3n+3)\cdot(3n+2)\cdot(3n+1)}$$
$$=\frac{27(1+1/n)^3|x|}{(3+3/n)\cdot(3+2/n)\cdot(3+1/n)}\longrightarrow\frac{27}{27}|x|=|x|$$

as $n\to\infty$. Setting this to be less than 1 we get $|x|<1$ and the sum converges (*at least*) on $(-1,1)$. Thus the radius of convergence is $\boxed{R=1}$.

12. (a) $a_n=\frac{1}{n^{-\ln n}}=n^{\ln n}=e^{\ln(n^{\ln n})}=e^{(\ln n)^2}$ *diverges* because $(\ln n)^2\to\infty$ as $n\to\infty$.

(b) $a_n=\sqrt{\frac{(1+n)n}{\sin n+n^2}}=\sqrt{\frac{(\frac{1}{n}+1)\cdot 1}{\frac{\sin n}{n^2}+1}}\longrightarrow\sqrt{\frac{0+1}{0+1}}=\sqrt{1}=\boxed{1}$ as $n\to\infty$ where we use the fact that $\frac{\sin n}{n^2}\to 0$ as $n\to\infty$ (you can use the *squeeze theorem* to prove this).

13. Use the identity

$$\sin^2 x=\frac{1}{2}(1-\cos 2x)$$

and plug in $y=2x$ into the known Taylor series for $\cos y$, and simplify algebraically in order to get $\boxed{1-\sin^2 x=1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n-1}}{(2n)!}x^{2n}}$.

Differential Equations

1. The ODE $y' = \cos^2 y \cdot \ln x$ is separable:

$$\frac{dy}{dx} = \cos^2 y \cdot \ln x \Rightarrow \frac{dy}{\cos^2 y} = \ln x \; dx \Rightarrow \int \sec^2 y \; dy = \int \ln x \; dx$$

$$\Rightarrow \tan y = \left\langle \begin{matrix} u & = & \ln x \\ du & = & \frac{dx}{x} \end{matrix} ; \begin{matrix} dv & = & dx \\ v & = & x \end{matrix} \right\rangle = \ln x \cdot x - \int x \cdot \frac{dx}{x}$$

$$= x \ln x - \int dx = x \ln x - x + C.$$

So we have $\tan y = x \ln x - x + C$ which implies: $\boxed{y = \tan^{-1}(x \ln x - x + C)}$.

2. $y' + \cos x \cdot y = \sin x \cdot \cos x$ is a linear first order ODE. The integrating factor is $I(x) = e^{\int \cos x \; dx} = e^{\sin x}$. Multiplying the differential equation on both sides of by the integrating factor, and using the "reverse" of the product rule one gets:

$$(e^{\sin x} y)' = e^{\sin x} \sin x \cdot \cos x \tag{1}$$

We want to integrate both sides, and to do this the hard part is integrating the right hand side.

$$\int e^{\sin x} \sin x \cdot \cos x \; dx = \langle t = \sin x \; ; \; dt = \cos x \; dx \rangle = \int e^t t \; dt$$

$$= \left\langle \begin{matrix} u & = & t \\ du & = & dt \end{matrix} ; \begin{matrix} dv & = & e^t \; dt \\ v & = & e^t \end{matrix} \right\rangle = te^t - \int e^t \; dt = te^t - e^t + C$$

$$= e^t(t-1) + C = e^{\sin x}(\sin x - 1) + C$$

Now integrating both sides of equation (1) and dividing by $e^{\sin x}$ we get: $\boxed{y = \sin x - 1 + Ce^{-\sin x}}$.

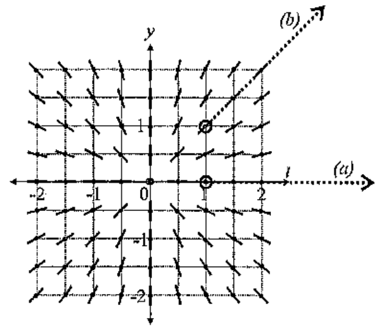
3. $y' = \frac{y}{x} + 2$ is a homogeneous first order ODE. So use the substitution $\langle v = \frac{y}{x} \; ; \; y' = v + xv' \rangle$ to get:

$$v + xv' = v + 2 \Rightarrow x\frac{dv}{dx} = 2 \Rightarrow dv = 2\frac{dx}{x}$$

$$\Rightarrow \int dv = 2 \int \frac{dx}{x} \Rightarrow v = 2 \ln |x| + C \Rightarrow \frac{y}{x} = 2 \ln |x| + C$$

$$\Rightarrow \boxed{y = 2x \ln |x| + Cx}$$

4. Notice that at any point, the direction field has as its slope its vertical coordinate divided by its horizontal coordinate. So at any point, the slope is the slope of the line between the point and the origin. Thus our direction field points radially away from the origin and we get the following picture:



5. **False**: Let's find the differential equation of each family and then check if the slopes are negative reciprocals of each other.

   - $x = ky^2$: Implicit differentiation with respect to x gives: $1 = k \cdot 2yy' \Rightarrow y' = \frac{1}{2ky}$. To find the differential equation, we need to replace the constant k by a function in x and y gotten from the original family of curves. In our case, $x = ky^2$ implies that $k = \frac{x}{y^2}$ so this is how we should replace k. Plugging this into $y' = \frac{1}{2ky}$ we get the family's differential equation:

   $$y' = \frac{1}{2y\frac{x}{y^2}} = \boxed{\frac{y}{2x}}.$$

   - $\frac{1}{2}x^2 + y^2 = c$: Implicit differentiation gives: $x + 2yy' = 0$ so this time we got rid of the constant c and we don't have to worry about replacing it, as in the first family. Solving for y' we get the differential equation:

   $$y' = \boxed{-\frac{x}{2y}}$$

   Apparently $\frac{y}{2x}$ is *not* the negative reciprocal of $-\frac{x}{2y}$ so that the two families are not orthogonal trajectories of each other.

   (Question: how could you change the families a bit to turn them into orthogonal trajectories?)

6. The characteristic equation of $y'' - 2y' - 3y = 0$ is $r^2 - 2r - 3 = (r-3)(r+1) = 0$ so $r = 3, -1$. Thus the general solution is $y = C_1e^{3x} + C_2e^{-x}$ whose derivative is $y' = 3C_1e^{3x} - C_2e^{-x}$. Plugging in the initial conditions we get: $3 = y(0) = C_1 + C_2$ and $1 = y'(0) = 3C_1 - C_2$. We want to solve for C_1 and C_2. For example, add the equations together to get $4C_1 = 4$ so $C_1 = 1$. Plug this into the first equation to get $C_2 = 2$. Thus the solution is: $\boxed{y = e^{3x} + 2e^{-x}}$.

7. The characteristic equation of $y''-2y'+5y=0$ is $r^2-2r+5=0$. Use the quadratic formula to get $r=1\pm 2i$. The complex solution gives a solution involving exponentials and sines/cosines. The real part of the root is 1 in our case implying that we'll have $e^{1\cdot x}=e^x$ as part of our solution. The imaginary part of the root is $\pm 2i$ meaning that we'll have sines/cosines of the form $\sin 2x, \cos 2x$. The solution is: $\boxed{y=e^x(C_1\cos 2x+C_2\sin 2x)}$

8. The characteristic equation of $y''-6y'+9y=0$ is $r^2-6r+9=(r-3)^2=0$ meaning that we have the *repeated* root $r=3$. e^{3x} will be a particular solution. Another *linearly independent* solution is gotten by multiplying the first solution by x to get xe^{3x}, giving us the general solution $y=e^{3x}(C_1+C_2x)$. Let's plug in the left boundary condition: $1=y(0)=(C_1+0)$ so $C_1=1$ and we can replace C_1 to get $y=e^{3x}(1+C_2x)$. Plug in the right hand boundary condition: $e^4+e^3=y(1)=e^3(1+C_2)$ so dividing by e^3 we get $e+1=1+C_2$ so that $C_2=e$ and our solution is: $\boxed{y=e^{3x}(1+ex)}$.

9. The best formula for the function $f(x,y)$ is given by choice (b) $\dfrac{e^y}{1+x^2}$. There are many ways to do this problem. A recommended method would be to use a process of elimination, and this can be done many ways. For example, notice that for our direction field, the slopes depend on the height, so the variable y *must* appear in the formula for $f(x,y)$. This eliminates (c) $\csc x$. Also, for our direction field, all slopes are positive, this eliminates (d) $\sin y$, as $\sin y$ is negative, for example, at $y=-0.5$. Finally, (a) cannot be the answer as the function $\dfrac{y}{x}+e^{\frac{y}{x}}$ is *not even defined* on the y-axis, since $x=0$ is being divided. This leaves (b) $\dfrac{e^y}{1+x^2}$ as the only possibility.

10. (a) Rewrite $y'\cdot\cos x = y\cdot\sin x+e^x\cos x$ as $\cos x\cdot y'-\sin x\cdot y=e^x\cos x$ so that we can use the *reverse* of the product rule to get:

$$(\cos x\cdot y)'=e^x\cos x$$

The *integrating factor* is just the function appearing next to the "y" after applying the reverse of the product rule. Therefore $I(x)=\cos x$ is the integrating factor and the answer is (v).

(b) The only difficult bit is integrating $e^x\cos x$. This is done by integrating by parts twice until the formula appears again and solving for the integral... One gets $\int e^x\cos x\,dx=\frac{e^x}{2}(\sin x+\cos x)+C$ So after integrating both sides of $(\cos x\cdot y)'=e^x\cos x$ and dividing both sides by $\cos x$ we get:

$$\boxed{y=\frac{e^x}{2}(\tan x+1)+C\sec x}$$

11. Use the method of *series solutions*:

$$y=\sum_{n=0}^{\infty}c_nx^n,\quad y'=\sum_{n=0}^{\infty}nc_nx^{n-1},\quad y''=\sum_{n=0}^{\infty}n(n-1)c_nx^{n-2}.$$

(Remember, it is possible *when necessary* to drop the first term of the sum for y' and the first two terms of y'', and re-index...) Continue:

$\frac{d^2y}{dx^2} = xy \Longrightarrow \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} = x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$. After dropping the first two terms of y'' and re-indexing it via $\langle n \mapsto n+3 \rangle$ we get:

$\sum_{n=-1}^{\infty} (n+3)(n+2)c_{n+3}x^{n+1} = \sum_{n=0}^{\infty} c_n x^{n+1}$

$\Longrightarrow 2c_2 + \sum_{n=0}^{\infty}[(n+3)(n+2)c_{n+3} - c_n]x^{n+1} = 0$. For this last equation to be true, each coefficient of x^n should be zero. That is, $c_2 = 0$ and the following recursion must also hold:

$$(n+3)(n+2)c_{n+3} = c_n, \quad n \geq 0.$$

Re-indexing by $\langle n \mapsto n-3 \rangle$ and solving for c_n we get:

$$c_n = \frac{1}{n(n-1)}c_{n-3}; \quad n \geq 3. \tag{2}$$

Use the initial conditions $y(0) = 1, \frac{dy}{dx}(0) = 0$ to conclude that $c_0 = 1$, and $c_1 = 0$. As the recursion skips by three, and $c_1 = c_2 = 0$, we infer that for n *not* a multiple of $3, c_n = 0$. This leaves us only needing to calculate c_{3k}. After applying equation 2 repeatedly, or finding $c_3, c_6, c_9 \ldots$ and looking for a pattern, one gets:

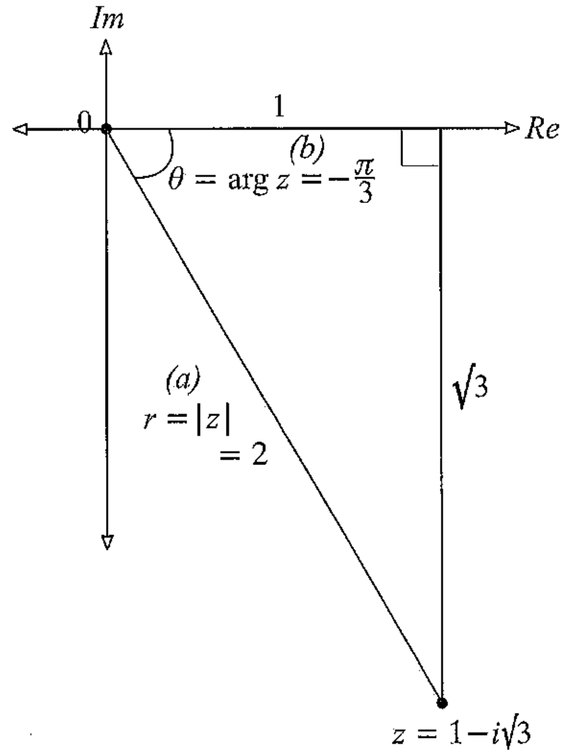
$$c_{3k} = \begin{cases} \frac{1}{(3k)(3k-1)\cdot(3k-3)(3k-4)\cdots 3\cdot 2}, & k > 0 \\ 1, & k = 0. \end{cases}$$

Thus we get the solution:

$$\boxed{y = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{(3k)(3k-1)\cdot(3k-3)(3k-4)\cdots 3\cdot 2}}.$$

Complex Numbers

1. It is very useful to construct a triangle in the complex plane to do this problem:



(a) If $z = a + ib$ then the *modulus* of z is defined by the Pythagorean-like formula: $|z| = \sqrt{a^2 + b^2}$ so that we get $|1 - i\sqrt{3}| = \sqrt{1+3} = \boxed{2}$.

(b) If $z = a + ib$ then the *argument* of z is defined by: $\tan(\arg z) = \dfrac{b}{a}$ so that we have $\tan(\arg(1 - i\sqrt{3})) = \dfrac{-\sqrt{3}}{1} = \dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$. So we look for an angle in the fourth quadrant (see the triangle above) whose tangent is $\dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$, e.g. $\arg z = \boxed{-\dfrac{\pi}{3}}$.

(c) Convert to polar form: $z = re^{i\theta} = 2e^{-i\frac{\pi}{3}}$. Then it is easy to raise this expression to the fifth power:

$$z^5 = (2e^{-i\frac{\pi}{3}})^5 = 2^5(e^{-i\frac{5\pi}{3}}).$$

The angle $-\dfrac{5\pi}{3}$ is the same as the angle $\dfrac{\pi}{3}$ so in terms of polar coordinates our expression becomes $32(e^{i\frac{\pi}{3}})$. Now use Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$ to convert back to rectangular coordinates:

$$z^5 = 32(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 32(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \boxed{16 + i16\sqrt{3}}.$$

2. (a) Use the quadratic formula to get: $\boxed{x = 1 \pm 2i}$

(b) For $x = 1 + 2i$:

$$x^2 = (1+2i)^2 = 1 + 2(2i) + (2i)^2 = 1 + 4i - 4 = \boxed{-3 + 4i}.$$

To find $\frac{1}{x}$ use the conjugate trick:

$$\frac{1}{x} = \frac{1}{1+2i} = \frac{1}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{1-2i}{1+4} = \boxed{\frac{1}{5} - \frac{2}{5}i}$$

Similarly, for $x = 1-2i$: $x^2 = \boxed{-3-4i}$ and $\frac{1}{x} = \boxed{\frac{1}{5} + \frac{2}{5}i}$.

3. (a) Solve $x^6 = -1$ as follows. Look for a solution of the polar form $x = re^{i\theta}$, and express -1 in polar coordinates but allow for the fact that every time you go 2π around a circle you get back where you started. That is, instead of saying $\arg(-1) = \pi$ we write $\arg(-1) = \pi + 2\pi k$ with k an integer. As $x^6 = (re^{i\theta})^6 = r^6 e^{6i\theta}$, our original equation $x^6 = -1$ becomes in polar notation:

$$r^6 \cdot e^{i6\theta} = 1 \cdot e^{i(\pi + 2\pi k)}.$$

Setting the moduli, and arguments equal separately we get:

$$r^6 = 1, \quad 6\theta = \pi + 2\pi k.$$

The first equation must be solved by a *non-negative* real number since absolute values are positive. This forces $r = 1$. The solution of the second equation is $\theta = \dfrac{\pi + 2\pi k}{6}$ for which we keep only the first six solutions (after that, we get back to where we started on the circle). So we get: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$, and $\frac{11\pi}{6}$. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to convert to rectangular coordinates we get: $\boxed{x = \frac{\sqrt{3}}{2} + i\frac{1}{2}, i, -\frac{\sqrt{3}}{2} + i\frac{1}{2}, -\frac{\sqrt{3}}{2} - i\frac{1}{2}, -i, \text{ and } -\frac{\sqrt{3}}{2} + i\frac{1}{2}}$.

(b) $x^6 + 1 = (x - x_1)(x - x_2)\cdots(x - x_6)$
$= \boxed{(x - \frac{\sqrt{3}}{2} - i\frac{1}{2})(x - i)(x + \frac{\sqrt{3}}{2} - i\frac{1}{2})(x + \frac{\sqrt{3}}{2} + i\frac{1}{2})(x + i)(x - \frac{\sqrt{3}}{2} + i\frac{1}{2})}$.

(c) To factor $x^6 + 1$ over $\mathbf{R}$, we need to pair up the above expression with complex-conjugates next to each other. Then use the fact that for any complex number a, with complex-conjugate $\overline{a}$ we have:

$$(x + a)(x + \overline{a}) = x^2 + (a + \overline{a})x + a\overline{a}$$

where $a + \overline{a}$ and $a\overline{a}$ are *real* numbers. Thus we get:
$x^6 + 1 = (x + i)(x - i) \cdot (x + \frac{\sqrt{3}}{2} + i\frac{1}{2})(x + \frac{\sqrt{3}}{2} - i\frac{1}{2}) \cdot (x - \frac{\sqrt{3}}{2} + i\frac{1}{2})(x - \frac{\sqrt{3}}{2} - i\frac{1}{2})$
$= \boxed{(x^2 + 1) \cdot (x^2 + x\sqrt{3} + 1) \cdot (x - x\sqrt{3} + 1)}$.